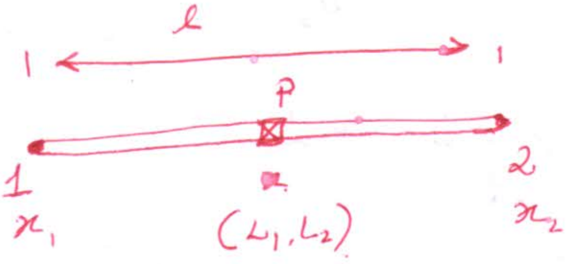
MODULE: 2

Natural Co-ordinates in One Dimension (L_1 & L_2)



Consider a two noded element as shown in figure. At any point P inside the ~~are~~ line element is identified by two natural Co-ordinates L_1 and L_2 and the Cartesian Co-ordinates x: ~~L_1 and L_2~~ Node 1 & Node 2 have the Cartesian Co-ordinates x_1 & x_2 respectively.

We know that
The total weightage of Natural co-ordinates at any point is unity

$$L_1 + L_2 = 1 \longrightarrow ①$$

At any point x within the element can be expressed as a linear combination of the nodal Co-ordinates of nodes 1 and 2 as

$$L_1 x_1 + L_2 x_2 = x \longrightarrow ②$$

arranging ① & ② in matrix form.

$$\begin{bmatrix} 1 & 1 \\ x_1 & x_2 \end{bmatrix} \begin{Bmatrix} L_1 \\ L_2 \end{Bmatrix} = \begin{Bmatrix} 1 \\ x \end{Bmatrix}$$

We need to find L_1 & L_2

$$\begin{bmatrix} 1 & 1 \\ x_1 & x_2 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 \\ x_1 & x_2 \end{bmatrix} \begin{Bmatrix} L_1 \\ L_2 \end{Bmatrix} = \begin{bmatrix} 1 & 1 \\ x_1 & x_2 \end{bmatrix}^{-1} \begin{Bmatrix} 1 \\ x \end{Bmatrix}$$

$\boxed{\text{Note: } A^{-1}A = AA^{-1} = I}$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} L_1 \\ L_2 \end{Bmatrix} = \begin{bmatrix} 1 & 1 \\ x_1 & x_2 \end{bmatrix}^{-1} \begin{Bmatrix} 1 \\ x \end{Bmatrix}$$

$$A^{-1} = \frac{1}{|A|} C^T$$

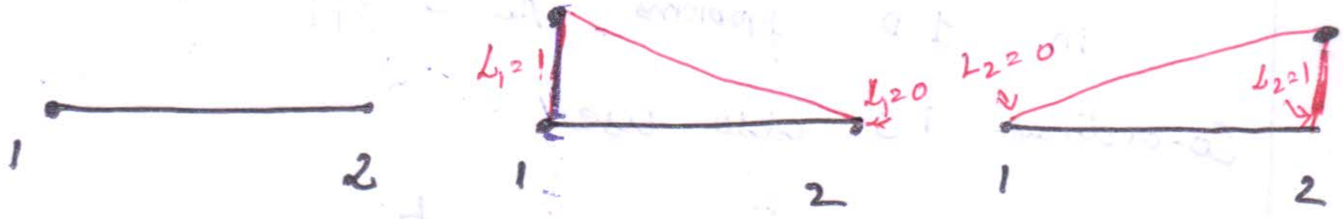
$$\begin{Bmatrix} L_1 \\ L_2 \end{Bmatrix} = \frac{1}{(x_2 - x_1)} \begin{bmatrix} x_2 & -1 \\ -x_1 & 1 \end{bmatrix} \begin{Bmatrix} 1 \\ x \end{Bmatrix}$$

$(x_2 - x_1 = l$ length of elem$)$

$$= \frac{1}{l} \begin{bmatrix} x_2 - x \\ -x_1 + x \end{bmatrix} = \frac{1}{l} \begin{bmatrix} x_2 - x \\ x - x_1 \end{bmatrix}$$

$$\begin{Bmatrix} L_1 \\ L_2 \end{Bmatrix} = \begin{Bmatrix} \dfrac{x_2 - x}{l} \\ \dfrac{x - x_1}{l} \end{Bmatrix}$$

The Variation of L_1 and L_2



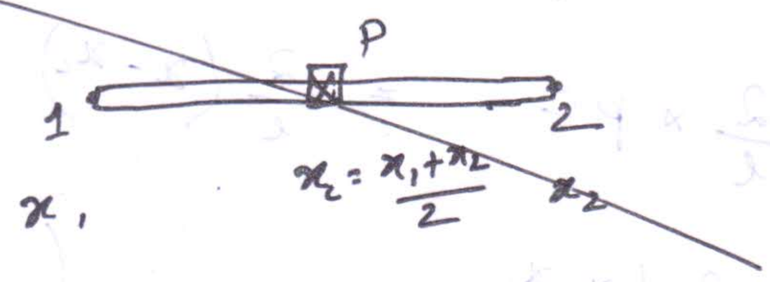
Element with node nos. — Variation of L_1 — Variation of L_2

The integration of polinomial terms in natural Co-ordinate can be performed by a formula.

$$\int_{x_1}^{x_2} (L_1)^{\alpha} (L_2)^{\beta} \cdot dx = \frac{\alpha!\,\beta!}{(\alpha+\beta+1)!} \times l_x.$$

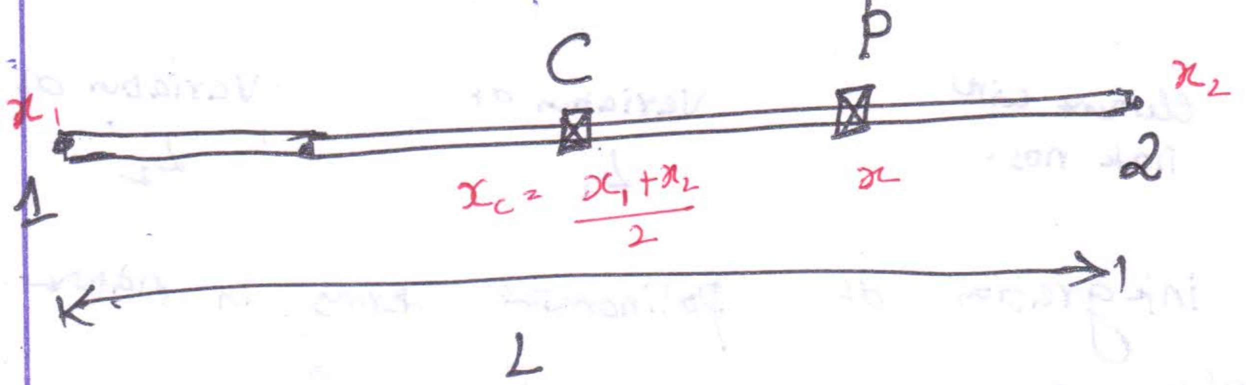
Natural Co-ordinate; ξ

in 1D 'ξ' type Natural Co-ordinate is also used.



Natural Co.ordinate 'ε'

in 1D problems, the 'ε' type natural co-ordinate is also used.



1 & 2 → Nodes, x_1, x_2 → Coordinates

C → Centre node. of node 1 & 2; P is the point reffer

The natural co-ordinate 'ε' for any point in the elements is defined as.

$$\xi = \frac{PC}{\left(\frac{x_2 - x_1}{2}\right)} = \left(\frac{PC}{l/2}\right) \quad (x_2 - x_1 = l)$$

$$= \frac{2}{l} * PC = \frac{2}{l}(x - x_c)$$

$$\xi \; \varepsilon = \frac{2}{l}(x - x_c)$$

$$= \frac{2}{l}\left[x - \frac{(x_1+x_2)}{2}\right]$$

$\left(\begin{array}{l} PC \\ \Rightarrow x - x_c \end{array}\right)$

$= \left(x - \frac{(x_1+x_2)}{2}\right)$

$$= \frac{2}{l}\left[x - \left(\frac{x_2 + x_1}{2}\right)\right]$$

adding + subtr x_1

$$= \frac{2}{l}\left[x - \left(\frac{x_2 + x_1 + x_1 - x_1}{2}\right)\right]$$

$$= \frac{2}{l}\left[x - \left(\frac{x_2 - x_1 + 2x_1}{2}\right)\right]$$

$$\xi = \frac{2}{l}\left[x - \left(\frac{l}{2} + x_1\right)\right] \qquad \boxed{(x - x_1 = l)}$$

$$\frac{\xi l}{2} = x - l/2 - x_1$$

$$\boxed{(\xi + 1)\frac{l}{2} = x - x_1}$$

Now applying the boundary condn

$x = x_1$ @ node = 1

$$l/2\,(\xi + 1) = 0$$

$$\xi = -1$$

$x = x_2$ at node 2

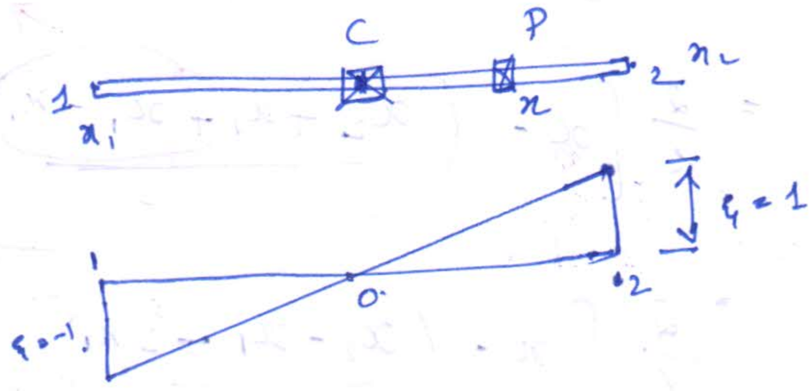
$$l/2\,(\xi + 1) = x_2 - x_1$$

$$\cancel{l}/2\,(\xi + 1) = \cancel{l}$$

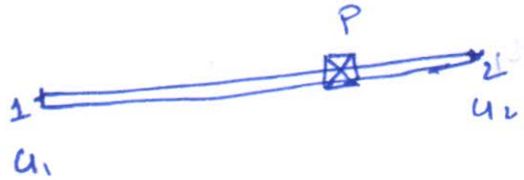
$$\xi = 2 - 1$$

$$\xi = 1$$

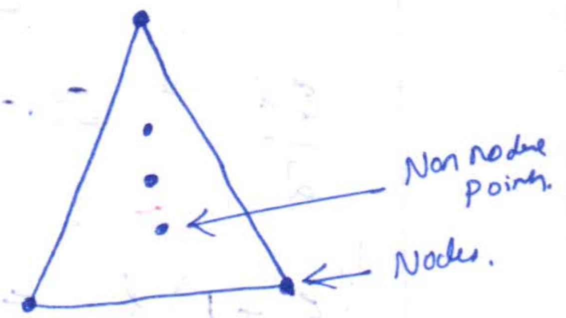
Variation of natural Co-ordinate ξ.



Shape functions:



1 D ..bar



2D. triangle elements.

The values of the field variable computed at the nodes are used to approximate the values at non-nodal points by interpolation of the nodal values.

eg. for triangle element :

* Nodes are exterior at any point within the elements. the field variable is described by the following approximate relation.

$$\phi(x,y) = N_1(x,y)\phi_1 + N_2(x,y)\phi_2 + N_3(x,y)\phi_3$$

ϕ_1, ϕ_2, ϕ_3 values of the field variables at node.

N_1, N_2, N_3 are the interpolation functions.
→ are also called as shape functions. because they are used to express the geometry or shape of the element.

Shape function has unit value at one nodal point and zero value at other nodal points.

for 1D., the basic field variable is displacement

$u = \sum N_i u_i \Rightarrow u \to$ displacement.

2 noded. $u = N_1 u_1 + N_2 u_2$

2D

$\Delta^{le} \Rightarrow u = N_1 u_1 + N_2 u_2 + N_3 u_3$
$v = N_1 v_1 + N_2 v_2 + N_3 v_3$

$\square^{le} \Rightarrow u = N_1 u_1 + N_2 u_2 + N_3 u_3 + N_4 u_4$
$v = N_1 v_1 + N_2 v_2 + N_3 v_3 + N_4 v_4.$

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* Characteristics and Condition to Satisfy Shape fn

* Refer PPT. (basic for mod 2 & 3)

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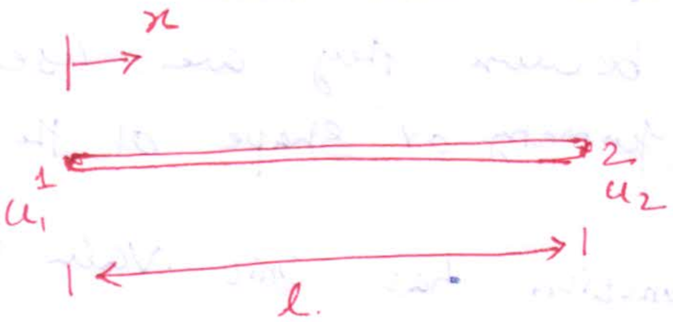
Derivation of the ~~bar~~ displacement function 'u' and shape function N for one dimensional bar element based on global co-ordinate approach.



Lets consider the above bar element with node 1 & 2

u_1 & u_2 → displacement @ respective nodes. (DOF of bar element)

from linear polynomial for 1D problems. we know

$$u = a_0 + a_1 x \rightarrow (1)$$

where a_0 and a_1 are global or generalized co-ordinates. ~~writing~~ Represent (1) in matrix form.

$$u = [1, x] \begin{Bmatrix} a_0 \\ a_1 \end{Bmatrix}$$

at node 1 we know that ∃

$u = u_1$, $x = 0$.

node 2: $u = u_2$, $x = l$.

Substituting the boundary condition in eqn (1).

$$\left.\begin{aligned} u_1 &= a_0 \\ u_2 &= a_0 + a_1 x \end{aligned}\right\} \rightarrow (2)$$

repren ② in matrx form..

$$\underbrace{\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}}_{u^*} = \underbrace{\begin{bmatrix} 1 & 0 \\ 1 & l \end{bmatrix}}_{[C]} \underbrace{\begin{Bmatrix} a_0 \\ a_1 \end{Bmatrix}}_{A} \rightarrow ?$$

$u^* =$ doF (~~displacement~~)

$[C]$ = Connecting matrx

$\{A\}$ = Generalised or global co-ordin matrx

$\cancel{u^*} =$

$$\begin{bmatrix} 1 & 0 \\ 1 & l \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 1 & l \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & l \end{bmatrix}^{-1} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$\begin{Bmatrix} a_0 \\ a_1 \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & l \end{bmatrix}^{-1} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$= \frac{1}{(l-0)} \begin{bmatrix} l & -1 \\ -0 & +1 \end{bmatrix}^{T} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$\begin{Bmatrix} a_0 \\ a_1 \end{Bmatrix} = \frac{1}{l} \begin{bmatrix} l & 0 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

substitn $\begin{Bmatrix} a_0 \\ a_1 \end{Bmatrix}$ in eqn ① we get

$$u = [1, x] \frac{1}{l} \begin{bmatrix} l & 0 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$= \frac{1}{l} [1, x] \begin{bmatrix} l & 0 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$= \frac{1}{l}\left[l - x \quad , 0 + x\right]\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$= \frac{1}{\cancel{l}}\left[\frac{l-x}{l} \; , \; \frac{0+x}{l}\right]\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$u = \left[\frac{l-x}{l} \; . \; \frac{x}{l}\right]\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$u = \left[N_1, \; N_2\right]\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$\boxed{N_1 = \frac{l-x}{l} \qquad N_2 = \frac{x}{l}}$$

$$u = N_1 u_1 + N_2 u_2$$

apply boundery Conditions:

at $N = 1$

$x = 0$

$N_1 = \frac{l}{l} = 1$

$N_2 = 0/l$

$= 0.$

$\boxed{N_1 = 1, \; N_2 = 0.}$

$N = 2$

$x = l.$

$N_1 = \frac{l - l}{l} = 0.$

$N_2 = l/l = 1$

$\boxed{N_2 = 1, \; N_1 = 0}$

* Satisfy the Condition of Shape function.